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THE ROLE OF THE ASYMMETRIC TERMS OF THE DIVERGENCE AND VORTICITY EQUATIONS IN THE ZERO LAPLACIAN VORTEX

by

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1.0   Introduction

A model of the axially symmetric vortex has been proposed by Nicholson and Johnson (1978), which describes the vortex in terms of its vorticity distribution as a function of radius. The vortex model is a composite of three lateral regimes in which the vorticity distribution is a piecewise smooth and continuous function of radius. This distribution along with a hypothesized divergence distribution is shown in Figure 1a, with the concurrent tangential and radial velocity distributions in 1b.

Nicholson and Johnson (1978) present data from hurricanes, tornadoes and waterspouts to demonstrate that the model is a good and reasonable approximation to the tangential wind above the boundary layer. This distribution satisfies Laplace's equation, hence the name, the zero Laplacian vortex. Examples of the fit of the velocity to the data are given below in Figures 2a and 2b.

Satisfaction of Laplace's equation by the vorticity distribution suggests that the vorticity may be part of an harmonic function (the imaginary part) and that the real part of the function, the divergence, also satisfies Laplace's equation, while observing matching boundary conditions.

This assumption implies that a distribution of divergence accompanies the vorticity distribution. Assuming, as a first approximation, that the atmosphere is horizontally incompressible, then from continuity the vertical velocity distribution may also be postulated.

Laplace's equation is a classic elliptic equation which has complex characteristics. The axially symmetric solution (linearity in $\ln r$) is not the only solution to Laplace's equation. The asymmetric solution is the product of an exponential and periodic function such that the Laplacian in Cartesian coordinates is given by

$$\nabla_2^2 a = \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} \tag{1}$$

then if

$$a = \gamma(n)e^{nx} \cos ny \text{ or}$$
$$a = \gamma(n)e^{ny} \cos nx \text{ or}$$
$$a = \gamma(n)e^{nx} \sin ny \text{ or}$$
$$a = \gamma(n)e^{ny} \sin nx \tag{2}$$

where n may be an integer in a Fourier series, and $\gamma(n)$ is a Fourier amplitude dependent on n,

$$\nabla_2^2 a = 0 \tag{3}$$

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The trick is to ascertain the relevance of the solution to the phenomena of the asymmetric vortex. This will be addressed in the following section.

2.0 Spiral Cloud Bands

It has long been established that the spiral cloud bands in hurricanes occur in the form of equiangular or logarithmic spirals. That is to say

$$\tan \alpha = \frac{r\partial\theta(r)}{\partial r} = \frac{\partial\theta(r)}{\partial\ln r} = \text{constant} \tag{4}$$

where α is the angle a log spiral makes with a ray emanating from the center of a polar coordinate system.

Since a log spiral satisfies (4) then it should plot in the semilog coordinates, θ, $\ln r$ as a straight line. Interestingly enough when multiple cloud bands are plotted in this coordinate system they not only form straight lines, but are parallel to one another and exhibit a certain amount of periodicity. Since the bands increase in intensity in a direction orthogonal to the periodicity one wonders whether they may exhibit behavior which would satisfy (2) and (3).

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In Cartesian coordinates it is axiomatic that rotation of coordinates implies a direct transformation of the form of the Laplacian from the old coordinate axes to the new ones (Figure 3). The Laplacian before rotation is expressed as

$$\nabla_2^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{5}$$

and after rotation as

$$\nabla_2^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} \tag{6}$$

In polar coordinates the Laplacian is written

$$\nabla_2^2 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \tag{7}$$

By multiplying the radial term by r/r, recombining and factoring out $1/r^2$ the Laplacian may be rewritten as

$$\nabla_2^2 = \frac{1}{r^2} \left(\frac{\partial^2}{\partial \ell n r^2} + \frac{\partial^2}{\partial \theta^2} \right) \tag{8}$$

Laplace's equation may dispense with the $1/r^2$ to give

$$r^2 \nabla_2^2 \, a = \left(\frac{\partial^2}{\partial \ell n r^2} + \frac{\partial^2}{\partial \theta^2} \right) a = 0 \tag{9}$$

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Thus, any linear solution in θ and ℓnr satisfies (9) as well as the product of an orthogonal set of exponential and periodic functions. As can be seen in (6) a rotation of the coordinate system θ, ℓnr by angle α in Figure 4 would give a new expression for Laplace's equation

$$r^2 \, \nabla_2^2 \, a \; = \; \left(\frac{\partial^2}{\partial S_r^2} \; + \; \frac{\partial^2}{\partial S_\theta^2} \right) \, a$$

where S_r and S_θ are the new coordinate axes illustrated in Figure 4. This rotation brings the cloud bands parallel to S_θ and periodic in S_r. Since both S_θ and S_r satisfy (4) they are log spirals, and the new coordinate system e^{S_θ} and e^{S_r} are log spiral coordinates. S_r and S_θ are in units of radians. They vary in the model from 0 to 2π, increasing to π and decreasing again to zero.

The scalar transformations between θ, ℓnr and S_θ, S_r are

$$S_r = \ell nr \, \cos\alpha \; + \; \theta\sin\alpha \tag{11a}$$

$$S_\theta = \theta\cos\alpha \; - \; \ell nr \, \sin\alpha \tag{11b}$$

Thus (2) may be rewritten as

$$a = \gamma(n)e^{n(S_\theta)} \cos(nS_r), \text{ etc.,}$$ being exponential in S_θ and periodic and linear in S_r.

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3. Spiral Space

3.1 Polar Coordinates

Spiral space is represented pictorially in Figure 5 below. The spirals are defined by S_r and S_θ being constants.

For

$$S_r = K = \ln r \, \cos\alpha + \sin\alpha$$

and

$$S_\theta = K' = \cos\alpha - \ln r \, \sin\alpha$$

the orthogonal log spirals which define spiral space are given by

$$r = \exp\left[\frac{K - \sin\alpha}{\cos\alpha}\right]$$

for constant S_r, i.e., the coordinate along which S_θ only changes and

$$r = \exp\left[\frac{\cos\alpha - K'}{\sin\alpha}\right]$$

for constant S_θ, i.e., the coordinate along which only S_r changes. Thus any point in spiral space may be defined in terms of values of the log spirals e^{S_r} and e^{S_θ}, or simply S_r and S_θ.

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3.2 Spherical Coordinates

Spiral space may also be defined in geographic or spherical coordinates, so that all algorithms applicable to polar coordinates may be applied to spherical ones as well. The two dimensional Laplacian in spherical coordinates ϕ, λ is given by

$$\nabla_2^2 = \frac{1}{r^2}\sin\phi \; \frac{\partial}{\partial\phi} \; \sin\phi \; \frac{\partial}{\partial\phi} + \frac{1}{r^2} \; \sin^2\phi \; \frac{\partial^2}{\partial\lambda^2}$$

rearranging terms gives

$$\nabla_2^2 = \frac{1}{r^2} \; \sin^2\phi \; \left[\frac{\partial}{\csc\phi\,\partial\phi} \; \frac{\partial}{\csc\phi\,\partial\phi} + \frac{\partial^2}{\partial\lambda^2} \right]$$

or

$$\nabla_2^2 = \frac{1}{r^2} \; \sin^2\phi \; \left[\frac{\partial^2}{\partial(\ln\,\tan(\frac{\phi}{2}))^2} + \frac{\partial^2}{\partial\lambda^2} \right]$$

Rotating the coordinate system $\ln\,\tan\frac{\phi}{2}$, λ through the angle α produces

$$\nabla_2^2 = \frac{1}{r^2} \; \sin^2\phi \; \left[\frac{\partial^2}{\partial S_\phi^2} + \frac{\partial^2}{\partial S_\lambda^2} \right]$$

where S_ϕ and S_λ are log spirals on a sphere conforming to the equation

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$$\tan \alpha = \frac{r \sin \phi \ \partial \lambda \ (\phi)}{\partial \phi} = \frac{r \ \partial \ \lambda(\phi)}{\partial \ln \tan(\frac{\phi}{2})} = \text{constant.}$$

The elliptic solution

$$e^{(S_\theta + i \ S_r)}$$

may be equally applied to

$$e^{(S_\lambda + i \ S_\theta)}$$

where

$$S_\phi = \ln(\tan \frac{\pi}{2}) \cos \alpha + \lambda \sin \alpha$$

and

$$S_\lambda = \lambda \cos \alpha - \ln(\tan \frac{\phi}{2}) \sin \alpha$$

The spherical space is defined in terms of spherical log spirals so that

$$\phi = 2 \tan^{-1} \left(\exp \left[\frac{K - \lambda \sin \alpha}{\cos \alpha} \right] \right)$$

for constant S_ϕ and

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$$\phi = 2 \tan^{-1} \left(\exp \left[\frac{\lambda \cos \alpha - K'}{\sin \alpha} \right] \right)$$

for constant S_λ .

For local spherical coordinates, where μ is the angle subtending the radial arc, and ν is the azimuthal angle the transformations between geographic ϕ , λ and local spherical μ , ν coordinates are given by

$$\lambda = \lambda_c + \tan^{-1} (\cos \nu / \cot \mu)$$

$$\phi = \phi_c + \cos^{-1} (\cos \mu / \cos (\lambda - \lambda_c))$$

and

$$\mu = \cos^{-1} (\cos \Delta\phi \cos \Delta\lambda)$$

$$\nu = \cos^{-1} (\cot \mu \tan \Delta\lambda)$$

where subscript "c" is the location of the center of the storm and the spiral coordinates are given by

$$S_\mu = \ln \tan \left(\frac{\mu}{2}\right) \cos \alpha + \nu \sin \alpha$$

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$$S_\nu = \nu \cos \alpha - \ln \tan \left(\frac{\mu}{2}\right) \sin \alpha$$

4. Transformations in Complex Space

The transformation from polar to spiral coordinates involves the successive transformations of shortening, rotation, scaling and stretching.

By considering complex space

$$Z = re^{i\theta} \qquad (12a)$$

and its conjugate

$$\bar{Z} = re^{-i\theta} \qquad (12b)$$

then by the transformation of shortening (operating on the conjugate in the northern hemisphere) we obtain

$$\ln\bar{z} = \ln r - i\theta \qquad (13)$$

Scaling by i and n where $i = \sqrt{-1}$ and n is an interger produces

$$ni \ln\bar{z} = n(\theta + i \ln r) \text{ (semilog coordinates)} \qquad (14)$$

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Rotation by the versor, $e^{i\alpha}$ (equivalent to rotating the semilog coordinates), produces

$$n i e^{i\alpha} \, \ln \bar{z} = n(\theta \pm \ln r) e^{i\alpha} = n(S_\theta + iS_r) \qquad (15)$$

Restretching to obtain real space curvilinear coordinates produces

$$\bar{z}^{\, n i e^{i\alpha}} = \exp \left[n(\theta + i \, \ln r) \, e^{i\alpha} \right] \qquad (16a)$$

$$= \exp \left[n(S_\theta + iS_r) \right] \qquad (16b)$$

The Laplacian in complex coordinates is given by

$$\nabla^2_2 = 4 \, \frac{\partial}{\partial z} \, \frac{\partial}{\partial \bar{z}} \qquad (17)$$

so that

$$4 \, \frac{\partial}{\partial z} \, \frac{\partial}{\partial \bar{z}} \, (\bar{z}^{\, n i e^{i\alpha}}) = 0 \qquad (18)$$

and 16a and b satisfy Laplace's equation.

Assuming the divergence and vorticity are the real and imaginary parts of an harmonic function, i.e., that

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they satisfy the Cauchy Riemann equations which in spiral
coordinates are

$$\frac{\partial \delta}{\partial S_r} = \frac{\partial \zeta}{\partial S_\theta} \ , \ \frac{\partial \zeta}{\partial S_r} = - \frac{\partial \delta}{\partial S_\theta} \tag{19}$$

then we may write the asymmetric solution for divergence
and vorticity as

$$\delta_n = e^{nS_\theta} \cos n \ S_r, \ \zeta_n = - e^{nS_\theta} \sin n \ S_r \tag{20}$$

where this solution is $\bar{F}$ where $F = \delta - i\zeta$ so that

$$Re(\bar{F}) = e^{n(S_\theta)} \cos S_r \tag{21}$$

and

$$Im(\bar{F}) = -e^{n(S_\theta)} \sin S_r$$

In this case the divergence field leads the
vorticity field by $\pi/2$. Thus the full solution for
symmetric and asymmetric distributions is

$$\delta = \delta_o + \iota_o \ell nr + \sum_{n=1}^{\infty} \gamma_o(n) e^{n(S_\theta)} \cos n \ S_r \tag{21a}$$

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$$\zeta = \zeta_0 + \iota_0 \ell n r - \sum_{n=1}^{\infty} \gamma_0(n) e^{n(S_\theta)} \sin(n \, S_r) \tag{21b}$$

where r is dimensionless, being understood as r/r_0 where $r_0 = 1L$. δ_0, ζ_0, ι_0 and γ_0 are constants of integration derived from boundary conditions.

5. Covariances in the Divergence and Vorticity Equations

In spiral coordinates the divergence and vorticity may be written:

$$\delta = \exp\left(-2(S_r \cos \alpha - S_\theta \sin \alpha)\right) \text{ x}$$

$$\left[\frac{\partial}{\partial S_r} \exp\left(S_r \cos\alpha - S_\theta \sin \alpha\right) U_{S_r} \right. \tag{22a}$$

$$\left. + \frac{\partial}{\partial S_\theta} \exp\left(S_r \cos\alpha - S_\theta \sin \alpha\right) U_{S_\theta} \right]$$

and

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$$\zeta = \exp(-2(S_r \cos\alpha - S_\theta \sin \alpha)) \ x$$

$$\left[\frac{\partial}{\partial S_r} \exp (S_r \cos\alpha - S_\theta \sin\alpha) \ U_{S_\theta} \right. \tag{22b}$$

$$\left. - \frac{\partial}{\partial S_\theta} \exp (S_r \cos\alpha - S_\theta \sin \alpha) \ U_{S_r} \right]$$

Since

$$\delta_n = \gamma_0(n) e^{n(S_\theta)} \cos nS_r$$

and

$$\zeta_n = \gamma_0(n) e^{n(S_\theta)} \sin n \ S_r$$

then by substitution and partial integration both U_{S_r} and U_{S_θ} have as part of their form

$$U_n \ \propto \ \int e^{ax} \cos pxdx \qquad \text{or}$$

$$U_n \ \propto \ \int e^{ax} \sin pxdx \tag{23}$$

which integrate to

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$$\frac{e^{ax}(a \cos px \quad p \sin px)}{a^2 + p^2} \qquad \text{or}$$

$$\frac{e^{ax}(a \sin px - p \cos px)}{a^2 + p^2} \qquad (24)$$

respectively. Hence both U_{S_r} and U_{S_θ} are out of phase by a non multiple of $\pi/2$ with the divergence and vorticity fields since in the velocity fields there exists the sum of sine and cosine terms while the divergence and vorticity fields are represented by sine and cosines respectively. (In Table I, found in the Appendix, the vorticity and divergence equations are listed term by term.)

As a first order approximation assume the vertical velocity, w, has the same periodic distribution as the negative divergence. The terms then may be viewed as products term by term of a Fourier series. If the terms are in phase then

$$\int_0^{2\pi} \sin^2\theta d\theta \ = \ \int_0^{2\pi} \cos^2\theta d\theta \ = \ \pi \qquad (25)$$

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and their azimuthal averages are non zero. If they are
out of phase then

$$\int_0^{2\pi} \sin\theta \cos\theta d\theta = 0 \tag{26}$$

and their azimuthal averages are zero, i.e., there exist
no positive and negative covariances from the asymmetric
terms which may contribute substantially to the hurricane
budget.

In spiral coordinates the integral over 2π
radians around the closed curve
of the real or imaginary part of the function

$$e^n(S_\theta + i S_r)$$

can be shown to be zero.

$$\text{Re} \left[e^{n(S_\theta + i S_r)} \right]$$

may be expressed by

$$e^{nS_\theta}\cos(n S_r).$$

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The integral over the closed curve above is given by

$$\int_0^{2\pi} \int_0^{2\pi} e^{nS_\theta} \cos(n\ S_r) r\ d\ S_r\ r\ d\ S_\theta \qquad (27)$$

In polar coordinates, where $\alpha = 0$, $r\ d\ S_r\ r\ d\ S_\theta$ reverts to $r\ d\ \ell n r\ r\ d\ \theta$, or $r\ d\ r\ d\ \theta$.

By substituting

$$r^2 = \exp 2(S_r \cos \alpha - S_\theta \sin \alpha)$$

into (27) we obtain

$$\int_0^{2\pi} \int_0^{2\pi} e^{nS_\theta} \cos(n\ S_r) e^{2\ S_r \cos\alpha} e^{-2\ S \sin\alpha} d\ S_r\ d\ S_\theta \qquad (28a)$$

$$\int_0^{2\pi} \int_0^{2\pi} e^{S_\theta (n-2 \sin \alpha)} e^{2S_r \cos\alpha} \cos(n\ S_r) dS_r dS_\theta \qquad (28b)$$

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Integrating S_θ between 0 and 2π we obtain

$$\frac{e^{2\pi(n-2\sin\alpha)}-e^{(n-2\sin\alpha)}}{n-2\sin\alpha} \int_0^{2\pi} e^{2S_r\cos\alpha} \cos(n\,S_r)dSr \qquad (29)$$

Letting $p = n$ and $a = 2\cos\alpha$ and $x = S_r$ and remembering that

$$\int e^{ax}\cos px\,dx - e^{ax}\frac{(a\cos px + p\sin px)}{a^2 + p^2} \qquad (30)$$

we obtain

$$\frac{e^{2\pi(n-2\sin\alpha)}-e^{(n-2\sin\alpha)}}{n-2\sin\alpha} \quad e^{2S_r\cos\alpha} \quad x$$

$$\frac{(2\cos\alpha\cos(nS_r) + n\sin(n\,S_r))^2}{4\cos^2\alpha + n^2} \left.\vphantom{\frac{1}{1}}\right|_0^{2\pi} = 0$$

The same procedure obtains for the imaginary part of the function. The importance of this fact is that azimuthally integrated terms of a simple Fourier series make no contribution to the overall storm budget. Unless the term

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is the product of a Fourier series with the terms in phase
as either the products of sines or cosines as indicated
above it makes no contribution.

Hence, if we look at the product of two terms,
such as occurs in $-\zeta\delta$ in the vorticity equation or ζ^2
in the divergence equation then we must consider the
products of the whole term, mean and Fourier components as
in (21a) and (21b). The investigation of the products of
terms A and B in the relevant equations gives us

$$(\bar{A} + A') \, (\bar{B} + B') = \bar{A}\,\bar{B} + \bar{A}\,B' + A'\bar{B} + A'B'$$

where $\bar{A}$ is the azimuthal average of A and A' is the
Fourier component. Both $\bar{A}B'$ and $A'\bar{B}$ vary in a simple
sinusoidal fashion. $\bar{A}\,\bar{B}$ does not vary at all, and
$A'B'$ varies as the product of sinusoidal functions.
Thus $\bar{A}\,B'$ and $A'\bar{B}$ contribute nothing to the azimuthally
integrated storm. $\bar{A}\,\bar{B}$, the product of the means, contri-
butes to the symmetric part and $A'B'$ the product of the
variances contributes to the asymmetric part.

This would at first appear contrary to customary
usage of perturbation theory. The underlying assumptions
here, of course, are that the asymmetries are not

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negligibly small in comparison to the symmetric part of the storm.

The asymmetric forms of the terms for the divergence and vorticity equations given in Table Al in the appendix are indicated in Tables I and II below, assuming that the specific volume varies as the divergence and the pressure varies as the vorticity.

It appears that as in the symmetric case, advective and tilting terms may balance one another, while the convective and enstrophy terms sum to be equal and opposite to the energy flux divergence. As opposed to the symmetric case, where the brake on storm development appears to be the frictional terms, in the asymmetric budget, the brake on the concentration of kinetic energy, the energy flux convergence arises out of the creation of divergence by enstrophy and the export of convergence by the convective terms. The advective and tilting terms appear uncoupled from the other terms. Possible subsets of equations in the asymmetric case of the divergence equation are listed below.

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$$-U_{S_{\theta}}\ \partial\zeta/\partial e^{S_r} = -\frac{\partial w}{\partial e^{S_{\theta}}}\ \frac{\partial U_{S_{\theta}}}{\partial Z}$$

and

$$-U_{S_r}\ \partial\zeta/\partial e^{S_{\theta}} = -\ \frac{\partial w}{\partial e^{S_r}}\ \frac{\partial U_{S_r}}{\partial Z}$$

Only the advective, twisting and solenoidal terms contribute.
They must sum to zero and be in phase for a steady state
storm.

The advective term creates convergence in most
of the storm.   The wind, $U_{S_{\theta}}$, must be *sufficiently high in*
order for the tilting term to balance it.   In the tropical
storm where U_{S_r} is trivial, even near the core, and
enstrophy and tilting are still at a low value compared
to the mature storm, the convective term may provide the
braking influence on the advective term.   Since the con-
vective term is related to the magnitude of the low level
convergence, a measure of storm intensity, the asymmetries
would need to be well developed for this balance to occur.
Otherwise, the advective term would lead the convective in
storm intensification.

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Again, let it be stressed that this is a highly
hypothetical, partial model of the two dimensional diver-
gence and vorticity fields in the tropical cyclone. Physi-
cal scenarios are suggested by a combination of phenomenology
and the mathematical possibility that both the divergence
and vorticity fields satisfy Laplace's equation in both
symmetric and asymmetric cases. Thus far it has only been
shown that the vorticity field satisfies the symmetric
version in a large majority of storms studied. Supporting
evidence for asymmetric satisfaction by divergence and
vorticity of Laplace's equation and symmetric satisfaction
by divergence, while indicative, is not yet firm enough to
warrant definite conclusions. Thus far, the model is
phenomenologically realistic and seems to be internally
consistent despite its elaborate ramifications.

The model may be tested numerically, using the
model as an analytic parameterization of a fully developed
numerical model including energy and thermodynamic consid-
erations. The partial analytical model here deals only
with the divergence and vorticity equations, forms of the
equations of motion, and a simple form of the continuity
equation. The model does not address the first law of
thermodynamics, the equation of state or the conservation
equation for the water substance. Further corroborating

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evidence may be obtained by a Fourier analysis of available spiral data of hurricanes from satellites such as the microwave detection of water droplet concentration in spiral rainbands studied by Allison, et al., (1974). A study of phenomena, such as water droplet content which may be correlated to the divergence field may prove to be indicative of the relevance of the mathematical solution to the actual processes of the storm.

6. Conclusion

The generation of divergence and to a lesser extent vorticity are highly dependent upon the development of asymmetries in this model. Because of the laborious mathematical analysis, a numerical model should be parameterized using the asymmetries of this model to assess their contributions to the heat, momentum and water vapor budgets of the hurricane.

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Table A.1 Comparison of Vorticity and Divergence Equations by Terms

	Vorticity $\frac{\partial \zeta}{\partial t} =$	Divergence $\frac{\partial \delta}{\partial t} =$
1)	Advective $-\underset{\sim}{U}\cdot\nabla\zeta$	Advective $-\underset{\sim}{U}\cdot\nabla\zeta\ x\underset{\sim}{k}$
2)	Convective $-w\,\frac{\partial\zeta}{\partial z}$	Convective $-w\,\frac{\partial\delta}{\partial z}$
3)	Twisting $-\nabla w\cdot\frac{\partial\underset{\sim}{U}}{\partial z}x\underset{\sim}{k}$	Tilting $-\nabla w\cdot\frac{\partial\underset{\sim}{U}}{\partial z}$
4)	Divergence $-\delta\zeta$	Enstrophy $+\zeta^2$
5)	Solenoidal $-\nabla\alpha x\nabla\rho\cdot\underset{\sim}{k}$	Toroidal $-\nabla\alpha\cdot\nabla\rho$
6)	Vertical mixing $\frac{\partial}{\partial z}\,\nu_z\,\frac{\partial\zeta}{\partial z}$	Vertical mixing $\frac{\partial}{\partial z}\,\nu_z\,\frac{\partial\delta}{\partial z}$
7)	Lateral mixing $\nu\nabla_2^2\zeta$	Lateral mixing $\nu\nabla_2^2\delta$
8)	-	Turbulent P $-\alpha\nabla^2 p$
9)	-	Energy dissipation $-\nabla^2\,(\underset{\sim}{U}\cdot\underset{\sim}{U}/2)$

APPENDIX A

A.1   The Role of Terms in the Axially Symmetric Vortex

A.1.1   Vorticity and Divergence Equations

In the axially symmetric vortex, the interactive terms of the vorticity and divergence equations come into play to reinforce the regime structure. These terms are listed and named in Table A.1 where both equations are presented vertically (read down) each term being named (∇ is the two dimensional gradient operator).

The first and third terms, which are difficult to visualize because they involve the cross product with the unit vertical vector, $\underset{\sim}{k}$, are illustrated in Figure A.1 for observed conditions under which they are positive and negative.

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A.1.2   The Characteristic Equations for Given Regimes

In the vortex model ordinarily there are more
unknowns than there are equations. This entails guessing
at the values of some of the unknowns in order to obtain a
consistent and coherent model. In this section we shall
see how we may increase the number of equations through
grouping analytically similar terms which should therefore
sum up in a self consistent way to satisfy the model.

Since the vortex is modeled in terms of vorticity
and divergence distribution the model gives an indication
of how the individual terms in the vorticity and divergence
equations behave. For the steady state vortex the terms
in scalar form appear to fall into discrete analytical
categories. The categorization is based on the terms having
similar analytical functions of radius and are of three
types. These are given in Table A.2. The categorization
is a function of regime and is given in Table A.3 for the
three regimes.

In Table A.3 the terms are listed by class for
the vorticity and divergence equations. The sign of the
term is given in parenthesis. In the core regime, the
Class II terms in the vorticity equation change sign from

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the "eye" portion of the core to the "eye wall" and hence
have a sign for each.

Assuming the coefficients of Class I sum to $Af(r)$,
Class II sum to $Bg(r)$ and Class III sum to $Ch(r)$ then the
steady state equation may be written:

$$Af(r) + Bg(r) + Ch(r) = 0 \qquad\qquad (A.1)$$

In the core C is zero since $\nabla^2_2 \zeta = -\nabla \alpha x \nabla p = 0$ so that

$$Af(r) + Bg(r) = 0 \qquad\qquad (A.2)$$

If A and B are non zero then $Af(r) = Bg(r)$. But this
means that $f(r) - -C_o g(r)$ where C_o is some other constant.
This may be true at one point but is not true elsewhere.
Thus, if A and B are zero $f(r) \neq C_o g(r)$ since $C_o = -B/A$
is undefined.

Assuming the parallel terms in the divergence
equations behave in the same fashion, then the terms
$-\nabla \alpha \cdot \nabla_p - \nabla^2 K$ must be of the form $Ch(r) = 0$.

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Therefore there exists a whole series of term couplings which shed light upon the vertical distribution of the terms. Thus, there exist the equations:

$$-\underset{\sim}{U}\cdot\nabla\zeta \overset{(-)}{-} \nabla w\cdot\frac{\partial U}{\partial z}\overset{(-)}{x}\underset{\sim}{k} + \frac{\partial}{\partial z}\overset{(+)}{\nu_z}\frac{\partial\zeta}{\partial z} = 0 \qquad (A.3)$$

and

$$-w\frac{\partial\zeta}{\partial z} - \delta\zeta = 0 \qquad (A.4)$$

Since we may solve (A.4) to obtain $\frac{\partial\zeta}{\partial z} = \zeta\delta/w$

and we know that $U_\theta = \frac{1}{2\pi}\int_0^r \zeta dr$ so that

$\frac{\partial U_\theta}{\partial z} = \frac{1}{2\pi}\int_0^r \frac{\partial\zeta}{\partial z} dr$ then we obtain by substituting (A.4)

into (A.3)

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$$-U_r \frac{\partial \zeta}{\partial r} - \frac{\partial w}{\partial r} \int_O^r \frac{\zeta \delta}{w} \, dr + \frac{\partial}{\partial z} \, \nu_z \, \frac{\partial \delta}{w} = 0 \qquad (A.5)$$

or

$$\nu_z = \frac{w}{\zeta \delta} \Big|_z \int_O^r U_r \frac{\partial \zeta}{\partial r} + \frac{\partial w}{\partial r} \int_O^r \frac{\zeta \delta}{w} \, dr \quad dz \qquad (A.6)$$

We therefore may obtain the value of the vertical coefficient
of eddy viscosity in the core.

By specifying the analytical form of the scalar
terms for a given regime we may solve for variables
otherwise unattainable such as the eddy viscosity, ν_z, as
a function of height and the specific volume, α, as a
function of radius. In this way we are able to diminish
the number of unknowns in the model because we have increased
the number of equations.

Appendix B

Creation of Regimes in the Vortex

B.1 Physical Parameters Governing the Vortex Structure

Perhaps the easiest way to understand the rationale for the lateral distribution of the regimes of the vortex model is to follow the suggested sequence of events which lead to the formation of the vortex.

In order to do this it is necessary to have an understanding of the role of the vorticity and divergence equations in the makeup of the model. Moreover, the role of the Richardson number is essential to understanding how order may be imparted to an otherwise amorphous structure.

B.2 Richardson Number

The Richardson number expresses the characteristic ratio of the energy of a work done against gravitational stability to energy transferred from mean to turbulent motion. Theoretical studies have placed the critical Richardson number from .25 to 2 with stability for higher values and instability for lower. The Richardson number may be expressed by

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$$R_i = g \frac{\partial \ln \theta}{\partial Z} \bigg/ \left(\frac{\partial U}{\partial Z}\right)^2 \qquad \text{or} \qquad R_i = g \frac{\partial p(\sigma, T)}{\partial Z} \bigg/ \left(\frac{\partial U}{\partial Z}\right)^2$$

$$\underbrace{\qquad\qquad\qquad}_{\text{atmospheric}} \qquad\qquad \underbrace{\qquad\qquad\qquad}_{\text{oceanic}}$$

Since air spiralling into the hurricane must
cross isobars to lower pressure the air experiences
cooling. Air within the cloud layer, however, cools
moist adiabatically, i.e., more slowly due to the release
of the latent heat of condensation of water vapor while
air below the cloud cools dry adiabatically. Since water
vapor liberates 597.3 calories per gram upon condensation,
the lower air cools much more rapidly than the upper. The
brake in the process, if the vortex is a hurricane, occurs
due to transfer of sensible heat from the sea surface
into the vortex boundary layer. The resulting stability
is only counteracted by increased turbulence due to a
higher mean velocity as one approaches the center of the
vortex. The more vigorous mixing tends to destroy the
inversion caused by the differential cooling. It is thus,
however, that the vertical velocity field tends to be
concentrated into a narrow throat.

The moist expansion in the cloud layer leads
to a higher potential temperature, θ of the air than its
lower neighbor. This is expressed by

$$\theta_e = \theta \ exp \left[\frac{Lw_s}{C_p T} \right]$$

where L is the latent heat of evaporation, 597.3 cals/gm,
w_s the saturation mixing ratio of the water vapor, C_p
the specific heat of dry air, and T the temperature of dry
air.

B.3 Sequence of Events in the Formation of a Severe
     Atmospheric Vortex

Given an arbitrary distribution of vorticity,
divergence and pressure, the following sequence of
events is hypothesized as leading to the formation of a
severe vortex.

Assume an initial parabolic distribution of pressure,
divergence and vorticity, with a subsequent radial and
tangential velocity distribution as shown in Figures
B1a and B1b. Assume also that the total amount of vor-
ticity and divergence are constant but may be and

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indeed are rearranged in response to the influence of the various terms of the relevant equations.

The first step in this process is the rearrangement of the distribution by the two dimensional Laplacians in the vorticity, divergence and pressure tendency equations. The distributions then become as illustrated in Figures B2a and B2b.

The second modification occurs when, in response to the concentration of the wind peaks inward, the inspiralling air experiences differential cooling. The extent of the cooling depends significantly upon whether the trajectory of the inspiralling air is within a cloud or below it. This differential cooling causes an inversion to form faster than turbulence can break it up in the outer regions of the vortex. This in turn suppresses vertical velocity making the formatiom of the inversion even more pronounced. Thus the critical Richardson number Ri defines the boundary between the inner and outer regimes (Figures B3a, B3b).

The final stage occurs when the enstrophy term, ζ^2, becomes so large at small values of radius (due to

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crowding of the vorticity and divergence into a smaller
and smaller area) that divergence is created so fast by
this term that the vorticity peak collapses like a
volcanic caldera bringing about the final stage in the
cycle illustrated in Figures B4a and B4b with the creation
of a core interior to the inner regime.

The fit of the model to the hurricane's pressure
profile is also very good and substantiates the presence
of the three regimes mentioned above. An extension of
the concept which accounts for the pressure and
tangential velocity fields presents a logical explanation
for the radial and vertical velocity fields.

The organization of the storm from a preexistent
but amorphous field of vorticity and convergence into the
highly structured vortex is seen as the effect of turbulent
lateral diffusion constrained by critical values of the
Richardson number and of enstrophy (the square of the
vorticity). The interaction of these mechanisms provides
a plausible scenario for the structuring of the mature

storm. The application of these concepts to the ocean,
changing what must be changed, will provide valuable
insights into the behavior of the upper ocean.

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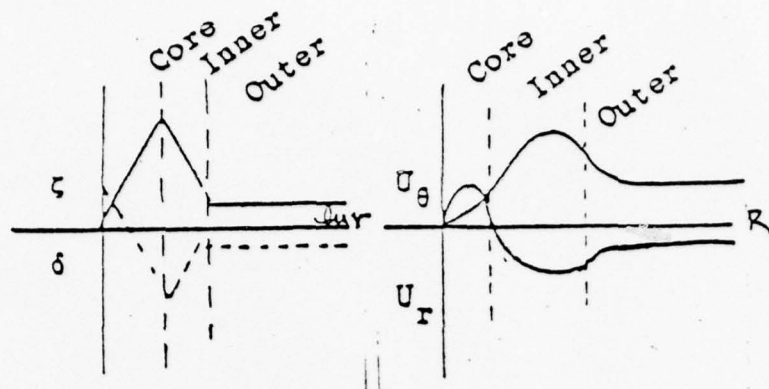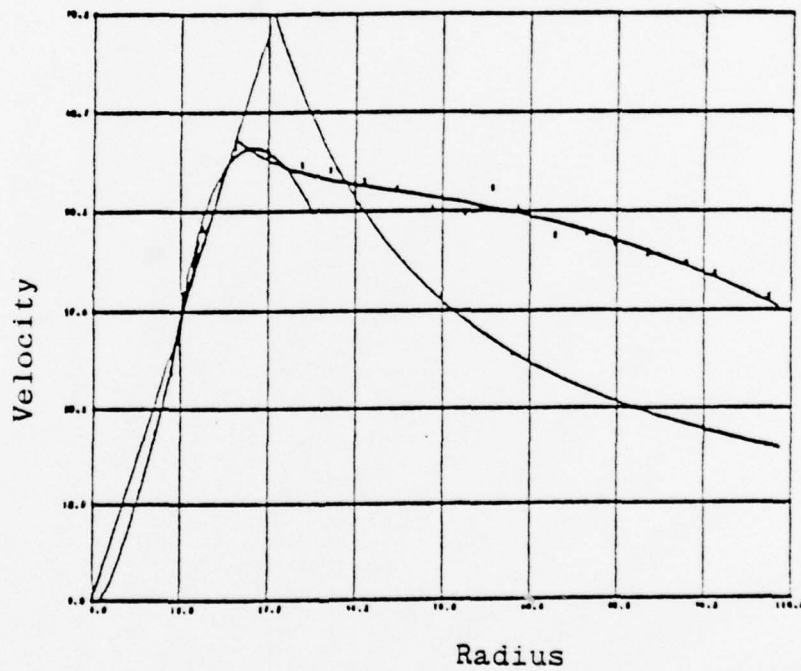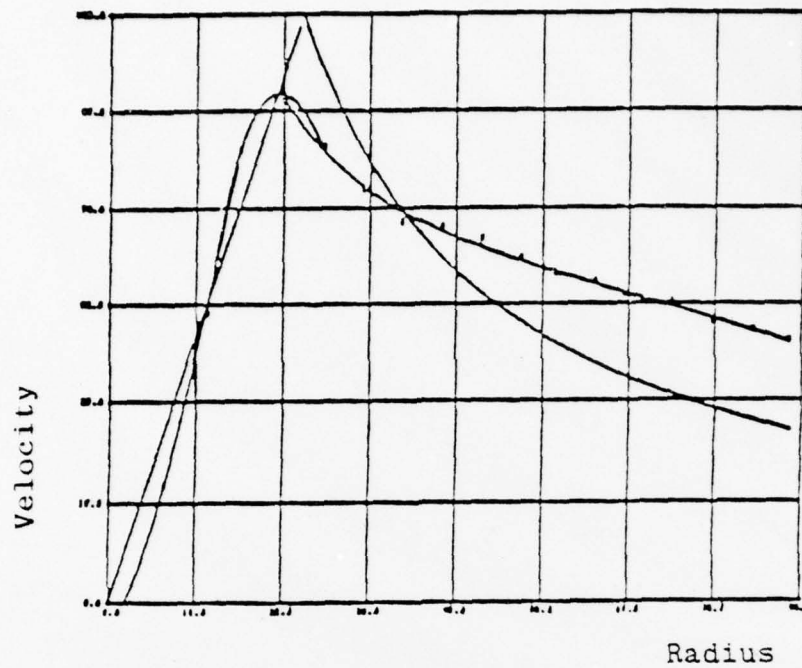   Reduction and Factorization, MWR,   104, pp. 641-649.

Figure 1a and 1b. Vorticity, divergence and velocity regimes in the zero Laplacian vortex.

Hurricane Cleo

Figure 2a. Piecewise continuous fit of Zero Laplacian Vortex to Riehl's (1963) hurricane Cleo data. Rankine's combined vortex is included for comparison.

Radius

Hurricane Hannah

Figure 2b. Piecewise continuous fit of Zero Laplacian Vortex to Riehl's (1963) hurricane Hannah data. Rankine's combined vortex is included for comparison.
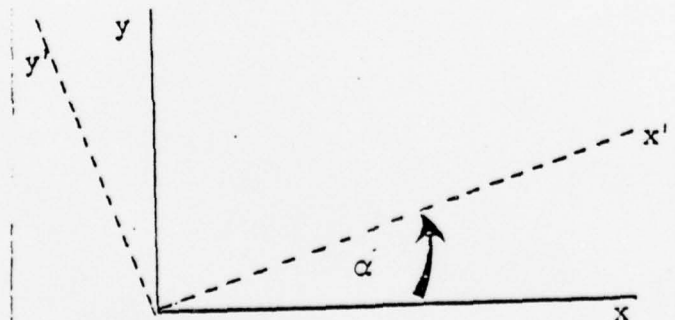

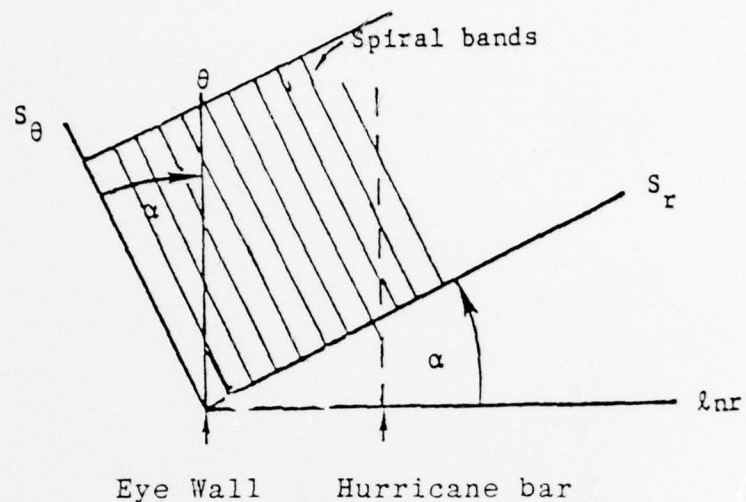Figure 3. Rotation of Cartesian Coordinate System through angle α.

Spiral bands

S_θ

θ

S_r

α

α

ℓnr

Eye Wall   Hurricane bar

Figure 4.   Rotation of semilog coordinate system
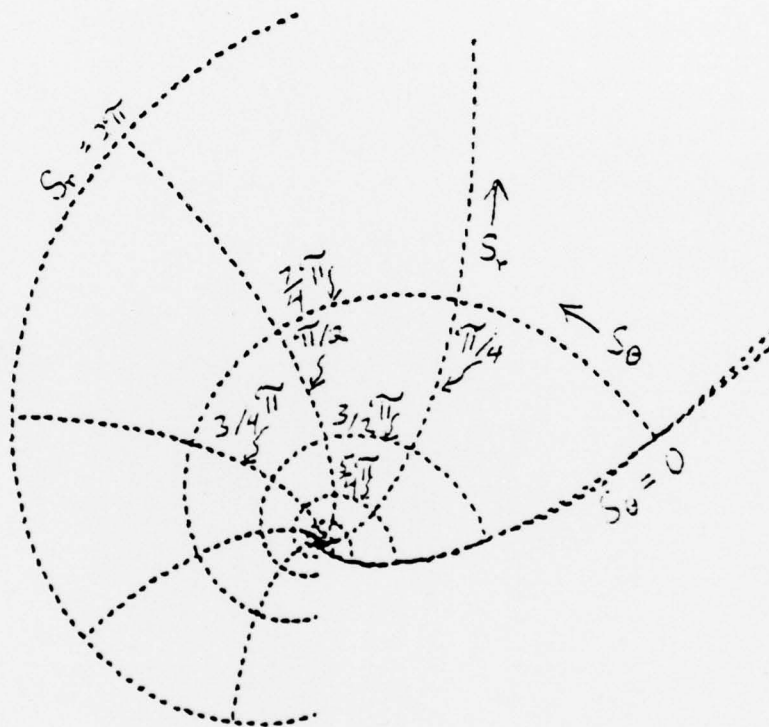ℓnr, θ to produce log spiral system
S_r, S_θ.

Figure 5.   Spiral Space with angle of rotation 20°.
The spirals occur at every $\pi/4$°. The
inner values of S_r would be completely
eclipsed by a strong axially symmetric
phenomenon at the storm center.